Language: English

Wednesday, May 10, 2023

Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$xf(x + f(y)) = (y - x)f(f(x)).$$

Problem 2. In triangle ABC, the incircle touches sides BC, CA, AB at D, E, F respectively. Assume there exists a point X on the line EF such that

$$\angle XBC = \angle XCB = 45^\circ.$$

Let M be the midpoint of the arc BC on the circumcircle of ABC not containing A. Prove that the line MD passes through E or F.

Problem 3. For each positive integer n, denote by $\omega(n)$ the number of distinct prime divisors of n (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever n is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$

Problem 4. Find the greatest integer $k \leq 2023$ for which the following holds: whenever Alice colours exactly k numbers of the set $\{1, 2, \ldots, 2023\}$ in red, Bob can colour some of the remaining uncoloured numbers in blue, such that the sum of the red numbers is the same as the sum of the blue numbers.

Time is 4 hours and 30 minutes

Each problem is worth 10 points