Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$xf(x + f(y)) = (y - x)f(f(x)).$$

Solution. Answer: For any real c, $f(x) = c - x$ for all $x \in \mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.

Let $P(x, y)$ denote the proposition that x and y satisfy the given equation. $P(0, 1)$ gives us $f(f(0)) = 0$.

From $P(x, x)$ we get that $xf(x+f(x)) = 0$ for all $x \in \mathbb{R}$, which together with $f(f(0)) = 0$ gives us $f(x + f(x)) = 0$ for all x.

Now let t be any real number such that $f(t) = 0$. If y is any number, we have from $P(t - f(y), y)$ the equality

$$(y + f(y) - t)f(f(t - f(y))) = 0$$

for all y and all t such that $f(t) = 0$. So, by taking $y = f(0)$ we obtain

$$(f(0) - t)f(f(t)) = 0 \quad \text{and hence} \quad (f(0) - t)f(0) = 0. \tag{A1-1}$$

Recall that as t with $f(t) = 0$ we can take $x + f(x)$ for any real number x. If for all reals x we have $x + f(x) = f(0)$, then $f(x)$ must be of the form $f(x) = c - x$ for some real c. It is straightforward that all functions of this form are indeed solutions.

Otherwise we can find some $a \neq 0$ so that $a + f(a) \neq f(0)$. If $t = a + f(a)$ in (A1-1), then $f(0)$ must be equal to 0. Now $P(x, 0)$ gives us $f(f(x)) = -f(x)$ for all real x. From here $P(x, x + f(x))$ gives us $xf(x) = -f(x)^2$ for all real x, which means for every x either $f(x) = 0$ or $f(x) = -x$.

Let us assume that in this case there is some $b \neq 0$ so that $f(b) = -b$. For any y we get from $P(b, y)$ and $f(f(b)) = -f(b) = b$ the equality $bf(b+f(y)) = (y-b)b$, which gives us $f(b+f(y)) = y-b$ for all y. If $y \neq b$, then the right hand side in the previous equality is not zero, so we must have $f(b+f(y)) = -b-f(y)$, which means that $-b-f(y) = y-b$, or that $f(y) = -y$ for all real y. But we already covered this solution (take $c = 0$ above). If there is no such b, then $f(x) = 0$ for all x, which gives us the final solution.

Thus, all such functions are of the form $c - x$ for real c or the zero function.

Problem 2. In triangle ABC, the incircle touches sides BC, CA, AB at D, E, F respectively. Assume there exists a point X on the line EF such that

$$\angle XBC = \angle XCB = 45^\circ.$$

Let M be the midpoint of the arc BC on the circumcircle of ABC not containing A. Prove that the line MD passes through E or F.

Solution 1. We first state a well-known lemma.

Lemma: In triangle ABC, let D, E, F be the points of tangency of the incircle to the sides BC, CA, AB and let I be the incenter. Then the intersection of EF and BI lies on the circle of diameter BC.

Proof: Let S be the intersection of EF and BI. $\angle BIC = \angle EFC$, hence S lies on the circumcircle of FIC in which IC is a diameter. Thus $\angle ISC = 90^\circ$, hence $\angle BSC = 90^\circ$.

Returning to the problem, let I be the incenter. The lemma implies that the two intersection points of EF with the circle of diameter BC are precisely the intersection points of EF with BI and CI. We have $\angle BXC = 90^\circ$, therefore either BX or CX is an internal angle bisector, which means either $\angle B = 90^\circ$ or $\angle C = 90^\circ$.

Assume, without loss of generality, that $\angle B = 90^\circ$. Then we have $\angle AMC = 90^\circ$, so M is the second intersection point of AI with the circle of diameter AC, thus the lemma implies that M lies on DF.

Solution 2.

Let I be the incenter of $\triangle ABC$ and let K be the foot of the perpendicular from D to EF. We begin by proving that $BKXC$ is cyclic, which can be done in two ways:

First Way. Note that $\angle KFD = 90^\circ - \dfrac{\angle C}{2}$ and $\angle KED = 90^\circ - \dfrac{\angle B}{2}$, so by using $KD \perp EF$, we have $\dfrac{FK}{ED} = \dfrac{\tan \frac{\angle C}{2}}{\tan \frac{\angle B}{2}}$. Similarly, since $\angle IBD = \dfrac{\angle B}{2}$ and $\angle ICD = \dfrac{\angle C}{2}$, by using $ID \perp BC$, we have $\dfrac{BF}{EC} = \dfrac{BD}{DC} = \dfrac{\tan \frac{\angle C}{2}}{\tan \frac{\angle B}{2}}$. Then, since $\angle BFK = 90^\circ + \dfrac{\angle A}{2} = \angle KEC$, we conclude that $\triangle BFK$ and $\triangle CEK$ are similar, so $\angle FKB = \angle CKE$ which shows line EF is the external-angle bisector of $\angle BKC$. Therefore, X lies on both the perpendicular bisector of the segment BC and the external angle bisector of $\angle BKC$ (and these lines are distinct) thus it lies on the circumcirle of $\triangle BKC$ (in particular the midpoint of arc BKC).

Second Way. Let T be the intersection of EF and BC, and N be the midpoint of the segment BC. It is well-known that $(T, D; B, C)$ is harmonic and $TB \cdot TC = TD \cdot TN$. On the other hand, since $XB = XC$, we have $\angle XND = 90^\circ = \angle XKD$, so $XKDN$ is cyclic and $TD \cdot TN = TK \cdot TX$. Therefore, we have $TK \cdot TX = TB \cdot TC$, which implies $BKXC$ is cyclic.

Now we will show that either $\angle B = 90^\circ$ or $\angle C = 90^\circ$. Note that $\angle BKC = \angle BXC = 90^\circ = \angle FKD = \angle EKD$ and $\angle FKB = \angle EKC$. Then we have

$$\angle FKB = \angle BKD = \angle DKC = \angle CKE = 45^\circ.$$

Hence BK bisects $\angle FKD$, but B also lies on the perpendicular bisector of DF. Therefore, either $FKDB$ is cyclic or $KF = KD$ while the former implies that $\angle B = 180^\circ - \angle FKD = 90^\circ$. In the latter case, we have $KB \perp FD$, which gives $90^\circ - \dfrac{\angle C}{2} = \angle KFD = 90^\circ - \angle FKB = 45^\circ$ and so $\angle C = 90^\circ$ as desired.

We consider, without loss of generality, the case where $\angle B = 90^\circ$. Observing that A, I, M are collinear we get:

$$\angle CDI = 90^\circ = \angle CBA = \angle CMA = \angle CMI$$

Hence $MDIC$ is cyclic so:

$$\angle MDC = \angle MIC = 180^\circ - \angle CIA = 180^\circ - \left(90^\circ + \frac{\angle B}{2}\right) = 45^\circ$$

We also have $\angle FDB = 90^\circ - \frac{\angle B}{2} = 45^\circ$ so $\angle FDB = \angle MDC$ and thus M, D, F are collinear as required.

Problem 3. For each positive integer n, denote by $\omega(n)$ the number of distinct prime divisors of n (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever n is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$

Solution. Answer: All polynomials of the form $f(x) = x^m$ for some $m \in \mathbb{Z}^+$ and $f(x) = c$ for some $c \in \mathbb{Z}^+$ with $\omega(c) \leq 2023^{2023} + 1$.

First of all we prove the following (well-known) Lemma. *Lemma:* Let $f(x)$ be a non-constant polynomial with integer coefficients. Then, the number of primes p such that $p|f(n)$ for some n is infinite.

Proof: If $f(0) = 0$, then the Lemma is obvious. Otherwise, define the polynomial

$$g(x) = \frac{f(xf(0))}{f(0)},$$

which has integer coefficients. Observe that $g(0) = 1$ and if g satisfies the property of the Lemma, then so does f. So, we need to prove that there are infinitely many primes p such that $p|g(n)$ for some n. Suppose, for the sake of contradiction that the number of such primes is finite, and let those be $p_1, ..., p_k$. Then, set $n = Np_1 \cdots p_k$ for some large N, such that $|g(n)| > 1$. It is evident that $g(n)$ has a prime divisor, but it is none of the p_i's. This is a contradiction and therefore the result follows.

Let $M = 2023^{2023} + 1$. Observe that constant polynomials $f(x) = c$ with $c \in \mathbb{N}$ such that $\omega(c) \leq M$ satisfy the conditions of the problem. On the other hand, if $f(x) = c$ with $\omega(c) > M$, we can choose some n such that $\omega(n) = M$ to see that the condition of the problem is not satisfied. Next, we look for non-constant polynomials that satisfy the conditions of the problem. Let $f(x) = x^m g(x)$, where $m \geq 0$ and $g(x)$ is a polynomial with $g(0) \neq 0$. We claim that g is a constant polynomial. Indeed, if it is not, then (due to the Lemma) there exist pairwise distinct primes $q_1, ..., q_{M+1}$ and non-zero integers $n_1, ..., n_{M+1}$ such that $q_i > |g(0)|$ and $q_i | g(n_i)$ for $i = 1, 2, ..., M+1$. Set $n = p_1 p_2 \cdots p_M$, where $p_1, ..., p_M$ are distinct primes such that

$$p_1 \equiv n_i \pmod{q_i}, \ \forall i = 1, 2, ..., M+1$$

and

$$p_j \equiv 1 \pmod{q_i}, \ \forall i = 1, 2, ..., M+1, \ \forall j = 2, 3, ..., M.$$

Observe that since $q_i > |g(0)|$, it is impossible to have $q_i | n_i$, so the existence of such primes is guaranteed by the Chinese Remainder Theorem and the Dirichlet's Theorem. Now, for every $i = 1, 2, ..., M+1$ we can see that $n = p_1 \cdots p_M \equiv n_i \pmod{q_i}$, which means that

$$g(n) \equiv g(n_i) \equiv 0 \pmod{q_i} \ \forall i = 1, 2, ..., M+1.$$

Thus, $\omega(f(n)) \geq \omega(g(n)) \geq M + 1 > M = \omega(n)$, which gives the desired contradiction. Therefore, $f(x) = cx^m$, for some $m \geq 1$ (since f was non-constant). If $c < 0$, take some n with $\omega(n) = M$ to see that $f(n)$ is negative and so, does not satisfy the conditions of the problem. If $c > 1$, choose some n with $\omega(n) = M$ and $\gcd(n, c) = 1$ to observe that f cannot satisfy the conditions of the problem. This means that $f(x) = x^m$ (which is for sure a solution to the problem) for some $m \geq 1$ and $f(x) = c$ for some $c \in \mathbb{Z}^+$ with $\omega(c) \leq M$ are the only polynomials that satisfy the conditions of the problem.

Problem 4. Find the greatest integer $k \leq 2023$ for which the following holds: whenever Alice colours exactly k numbers of the set $\{1, 2, \ldots, 2023\}$ in red, Bob can colour some of the remaining uncoloured numbers in blue, such that the sum of the red numbers is the same as the sum of the blue numbers.

Solution. Answer: 592.

For $k \geq 593$, Alice can color the greatest 593 numbers $1431, 1432, \ldots, 2023$ and any other $(k-593)$ numbers so that their sum s would satisfy

$$s \geq \frac{2023 \cdot 2024}{2} - \frac{1430 \cdot 1431}{2} > \frac{1}{2} \cdot \left(\frac{2023 \cdot 2024}{2}\right),$$

thus anyhow Bob chooses his numbers, the sum of his numbers will be less than Alice's numbers' sum.

We now show that $k = 592$ satisfies the condition. Let s be the sum of Alice's 592 numbers; note that $s < \frac{1}{2} \cdot \left(\frac{2023 \cdot 2024}{2}\right)$. Below is a strategy for Bob to find some of the remaining 1431 numbers so that their sum is

$$s_0 = \min\left\{s, \frac{2023.2024}{2} - 2s\right\} \leqslant \frac{1}{3} \cdot \left(\frac{2023 \cdot 2024}{2}\right),$$

(Clearly, if Bob finds some numbers whose sum is $\frac{2023.2024}{2} - 2s$, then the sum of remaining numbers will be s).

Case 1. $s_0 \geq 2024$. Let $s_0 = 2024a + b$, where $0 \leqslant b \leqslant 2023$. Bob finds two of the remaining numbers with sum b or $2024 + b$, then he finds a (or $a - 1$) pairs among the remaining numbers with sum 2024. Note that $a \leq 337$ since $s_0 \leq \frac{1}{3} \cdot \left(\frac{2023 \cdot 2024}{2}\right)$.

The $\left\lfloor \frac{b-1}{2} \right\rfloor$ pairs

$$(1, b-1),\ (2, b-2),\ \ldots,\ \left(\left\lfloor \frac{b-1}{2} \right\rfloor, b - \left\lfloor \frac{b-1}{2} \right\rfloor\right),$$

have sum of their components equal to b and the $\left\lfloor \frac{2023+b}{2} \right\rfloor - b$ pairs

$$(2023, b+1),\ (2022, b+2),\ \ldots,\ \left(2024 + b - \left\lfloor \frac{2023+b}{2} \right\rfloor, \left\lfloor \frac{2023+b}{2} \right\rfloor\right)$$

have sum of their components equal to $2024 + b$. The total number of these pairs is

$$\left\lfloor \frac{2023+b}{2} \right\rfloor - b + \left\lfloor \frac{b-1}{2} \right\rfloor \geq \frac{2022+b}{2} + \frac{b-2}{2} - b = \frac{2020}{2} = 1010 > 592,$$

hence some of these pairs have no red-colored components, so Bob can choose one of these pairs and color those two numbers in blue. Thus 594 numbers are colored so far.

Further, the 1011 pairs

$$(1, 2023),\ (2, 2022),\ \ldots,\ (1011, 1013)$$

have sum of the components equal to 2024. Among these, at least $1011 - 594 = 417 > 337 \geq a$ pairs have no components colored, so Bob can choose a (or $a-1$) uncolored pairs and color them all blue to achieve a collection of blue numbers with their sum equal to s_0.

Case 2. $s_0 \leq 2023$. Note that $s \geq 1 + 2 + \ldots + 592 > 2023$, thus we have $s_0 = \frac{2023 \cdot 2024}{2} - 2s$, i.e. $s = \frac{2023 \cdot 2024}{4} - \frac{s_0}{2}$.

If $s_0 > 2 \cdot 593$, at least one of the 593 pairs

$$(1, s_0 - 1),\ (2, s_0 - 2),\ \ldots,\ (593, s_0 - 593)$$

have no red-colored components, so Bob can choose these two numbers and immediately achieve the sum of s_0. And if $s_0 \leq 2 \cdot 593$, then

$$s = \frac{2023 \cdot 2024}{4} - \frac{s_0}{2} \geq (1432 + 1433 + \ldots + 2023) - 593 = 839 + (1434 + 1435 + \ldots + 2023),$$

hence Alice cannot have colored any of the numbers $1, 2, \ldots, 838$. Then Bob can easily choose one or two of these numbers having the sum of s_0.